Grade 7/8 Math Circles
February 5 & 6 & 7 & 8, 2024
The Pigeonhole Principle

Introduction

Did you know that there are two people in Toronto who have the exact same number of hairs on their head? We can determine this because there are more people in Toronto than there are hairs on a human head. To conclude this, we are using what is known as the Pigeonhole Principle.

A pigeonhole is a small cubbyhole for storing letters or other items. (The name comes from similar structures used to house pigeons.)

Figure 1: A pigeonhole message box[1] for storing mail.

Pigeonholes are a way to sort things into many different categories. For example, we may want to sort the students in a class by their birth month or put them into one of four teams.

Notice that if there are more than 12 students in the class, then one of the months must be the birth month of at least two students. If there are more than 4 students in the class, then one of the teams must have at least two students. This principle holds no matter what the categories or what the items that go into the categories are.

The Pigeonhole Principle

If there are less categories than items that go into those categories, then one category must contain at least two items.

[1]**Source:** Stacalusa, CC0, via Wikimedia Commons

Let's start with an example that requires us to first determine the number of categories, so that we can figure out how many items going into the categories will guarantee that one category contains two or more items.

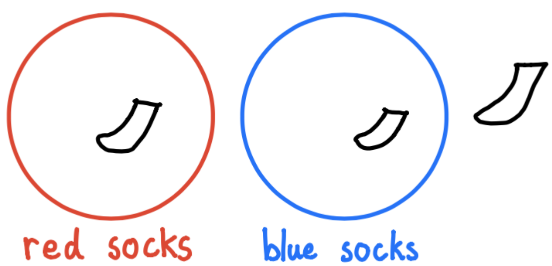
Example 1

Suppose that I have a bag that contains 50 red socks and 50 blue socks. Imagine that I pull out socks without looking to find a pair with the same colour. How many socks do I have to pull out in order to be sure that there is such a pair?

Let's use the Pigeonhole Principle to solve this.

The Pigeonhole Principle	If there are less categories than items that go into those categories, then one category must contain at least 2 items.

In this example, there are 2 categories (red socks and blue socks).



If there are any more socks than categories, then one one category must contain 2 socks, which would be a matching pair.

Therefore, we would have to pull out just $2+1=3$ socks to guarantee a pair.

Activity

To see the Pigeonhole Principle in action, try making a conclusion about the number of people in this Math Circles session born on the same day of the month. See whether the guess that you make is accurate.

# of people in this Math Circles session: ______

# of days in the longest month: ______

Predict whether two people are born on the same day of the month: ______

Were two people born on the same day of the month? Yes No

Make a conclusion about what you have learned about the attendees in the session. Be sure to cite the Pigeonhole Principle.

Now we will split the session into smaller groups of students. Within this smaller group, see if you can use the Pigeonhole Principle again to form a conclusion. Complete the following information.

# of people in my group: ______

# of months in a year: ______

Predict whether two people are born in the same month: Yes No

Were two people born on the same day of the month? Yes No

Make a conclusion about what you have learned about the people in your group. Be sure to cite the Pigeonhole Principle.

Examples

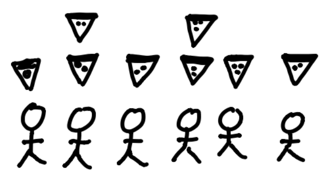
Let's look now at more examples of this principle being applied. For each example, ask these questions: How do we know? What are the categories and what are the items that we are putting into each category?

Example 2: A pizza is cut into 8 slices and shared between 6 friends. If everyone eats whole slices only, at least one person would get at least two slices.

Example 3: A flock of pigeons land on some trees. There are more pigeons than trees. After all the pigeons have landed, at least one of the trees contains more than one pigeon.

Example 4: If there are 5 classes and 4 teachers to teach those classes, then at least one of the teachers will teach more than one class.

Example 2: The categories are the 6 friends.
The items are the 8 slices of pizza.



There are more slices of pizza than friends.
By the Pigeonhole Principle, at least one of the friends receives more th
one pizza.

Example 3: The categories are the pigeons.
The items are the trees.

There are more pigeons than trees.
By the Pigeonhole Principle, one tree has 2 birds.

Example 4: The categories are the 4 teachers.
The items are the 5 classes.

There are more classes than teachers.
By the Pigeonhole Principle, one teacher must teach 2 classes.

In all cases, #categories < #items.
So by the Pigeonhole Principle, one category contains at least 2 items.

Example 5: In a room of 367 people (such as a packed auditorium), there are always two people who share the same birthday.

Example 6: In this Math Circles session, two students are born on the same day of the month (see Activity on page 3).

Example 5: The categories are the days of the year.
(There are 365 days in a non-leap year and 366 days in a leap year.)
The items are the people in the room.

There are more people in the room than days in any year.
By the Pigeonhole Principle, 2 people in the room share a birthday.

Example 6: There are 31 possible days of the month.
But there are more than 31 students in this Math Circles session.

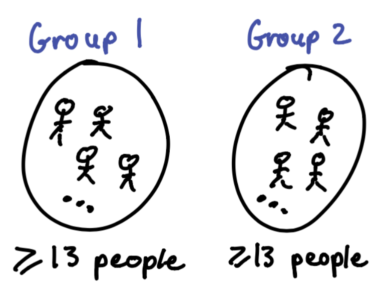
By the Pigeonhole Principle, at least 2 students are born on the same day of the month.

Example 7: Split the students in our Math Circles session into two groups that have about the same number of people. In each group, at least two students are born in the same month (see Activity on page 3).

Example 8: For any consecutive sequence of 27 words (excluding numbers) on this page, at least two of them will begin with the same letter.

Example 7: When we split the students into 2 roughly equal groups, there are at least 13 students in each group. But there are only 12 months in a year.

By the Pigeonhole Principle, at least 2 students are born in the same month.

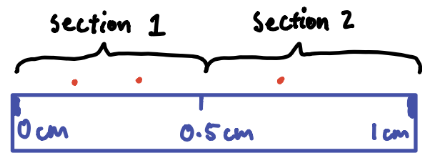


Example 8: There are 26 letters in the English alphabet.
So in a sequence of 27 English words, at least 2 must begin with the same letter.

Example 9: Three points are randomly placed on a ruler between 0 cm and 1 cm. At least two of the points are within 0.5 cm of each other.

Example 10: Choose 11 different integers from $1, \ldots, 99$. There are always two numbers whose difference (the answer when subtracting the smaller number from the larger number) is less than 10.

Example 9: Divide the ruler into 2 sections:
1) from 0 cm to 0.5 cm
2) from 0.5 cm to 1 cm



There are 3 points that lie in one of the 2 sections.

By the Pigeonhole Principle, one of the sections must contain at least 2 points. These 2 points would then be within 0.5 cm of each other.

Example 10: We can create the following categories:
1) the integers from 1 to 9
2) the integers from 10 to 19
3) the integers from 20 to 29
4) the integers from 30 to 39
5) the integers from 40 to 49
6) the integers from 50 to 59
7) the integers from 60 to 69
8) the integers from 70 to 79
9) the integers from 80 to 89
10) the integers from 90 to 99

If we choose 11 integers (from 1 to 99), then the 11 numbers each belong to one of the 10 categories above.

By the Pigeonhole Principle, there are two numbers that are into the same category. These two numbers would be within 9 whole numbers away from each other.

Example 11: (*) Given a list of $n+1$ different integers, there are two integers in the list which differ by a multiple of n. (For example, since there are at least 2025 powers of two, this means that we can find two different powers of two that differ by a multiple of 2024.)

Example 12: (**) A *repunit* is a number consisting entirely of the digit 1. For example, the numbers 1, 11, and 1111111 are repunits. There is a repunit that is divisible by 2023.

Note: Examples marked with asterisks require the application of other areas of mathematics such as prime numbers and divisibility.

Example 11: STRATEGY Find the remainder upon division by n of each of the n+1 numbers in the list.

When dividing by n, there are only n distinct remainders possible.
However, there are n+1 numbers on the list.
By the Pigeonhole Principle, 2 of the numbers on the list leave the same remainder upon division by n. These two numbers are a multiple of n apart from each other.

Example 12: There are at least 2024 repunits.
From Example 11, there are 2 repunits that differ by 2023.
So the difference of these repunits is divisible by 2023.

Their difference looks like some 1's followed by some 0's.

$$\begin{array}{r} 11\ldots111\ldots1 \\ -\quad 11\ldots1 \\ \hline \underbrace{11\ldots1}_{x}\underbrace{00\ldots0}_{\times 10\times10\times\ldots\times10} \end{array}$$

(divisible by 2023)

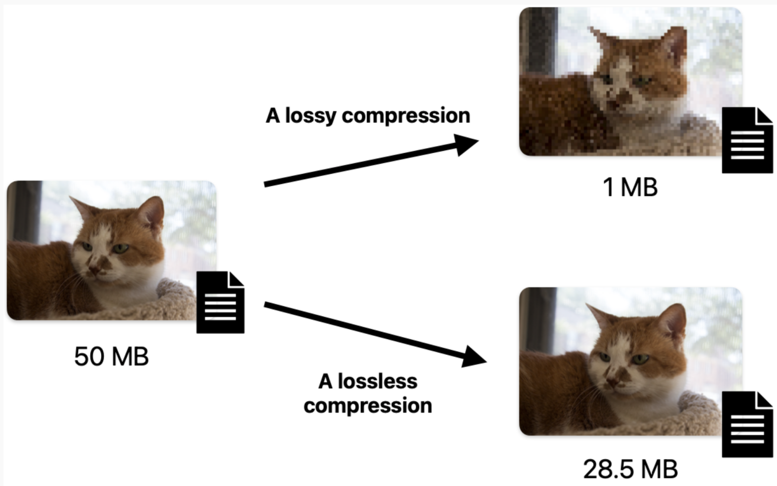
We can factor the difference into a repunit x multiplied by some number of 10's.

none of these factors are 10 (or 2 or 5)

Since $2023 = 7\times17\times17$ and $x\times10\times10\times\ldots\times10$ is divisible by 2023, then x must be divisible by 2023.

Case Study: Compression Algorithms

Compression algorithms turn large files into smaller files. For example, an image that takes up 1 MB (million bytes) of space may be compressed to take up only 1 kB (thousand bytes) of space. If the original file can be recovered perfectly from the compressed file, then the compression is said to be lossless.



Suppose that the designer of a compression algorithm promotes their product with the following claim:

We can compress any size data to under 1 MB completely lossless. The file type does not matter.

Example: we can compress any 1 TB file to under 50 kB in under just 10 seconds and decompress it in under the same amount of time.

This is done using advanced mathematics that has not been discovered until 6 months ago.

Think carefully. What is questionable about this claim?

Answer. There are many more 1 TB files than there are files that are 50 kB or smaller. This is because a 1 TB file stores 1,000,000,000,000 bytes of information. On the other hand, a 50 kB stores only 50, 000 bytes of information. (A byte can have one of 256 different values.)

So two 1 TB files must be compressed into the same smaller file, in which case the compression is not lossless, or else some 1 TB files become larger after the compression algorithm is run. In general, any lossless compression algorithm will turn some files larger than they originally were.

By the Pigeonhole Principle, it is not possible for a compression algorithm to transform each 1 TB file into a different file that is 50 KB or smaller.

The Crowded Pigeonhole Principle

In Example 5, we observed that a room of 367 people must contain two people who share the same birthday. There are at most 366 days in a year (because of leap years). This is less than the people in the room, and so the Pigeonhole Principle tells us that two people must have the same birthday.

What if there are more people than twice the number of birthdays? Twice the number of birthdays is $2 \times 366 = 732$. What if there are 733 people?

Key fact: If there are 733 people, then one of the birthdays must be shared by three or more people.

Why is this true? Let's suppose that it's not. So each day of the year is the birthday of two different people at most. Then the maximum number of people is the number of days in the longest year multiplied by the maximum number of people who share the same birthday: $2 \times 366 = 732$. This is impossible since there are 733 people in the room! So there has to be one day that is the birthday for at least 3 people.

Let's extend the Pigeonhole Principle to account for situations in which there are more items than n times the number of categories.

> **The Crowded Pigeonhole Principle (aka Pigeonhole Principle for Multiple Repeated Items)**
>
> If there are more items than n times the number of categories, then one category must contain at least $n + 1$ items.
>
> **Also:** If there are c categories and k items that go into those categories, then we can guarantee that at least $\lceil \frac{k}{c} \rceil$ items are in the same category.

(The symbol $\lceil\ \rceil$ means to round up, which is also known as "taking the ceiling of a number.")

In Example 1, we've already seen that, in a bag of 50 red socks and 50 blue socks, we only need to pull out three socks to guarantee that two of the socks have the same colour. What if we wanted to guarantee that three of the socks have the same colour?

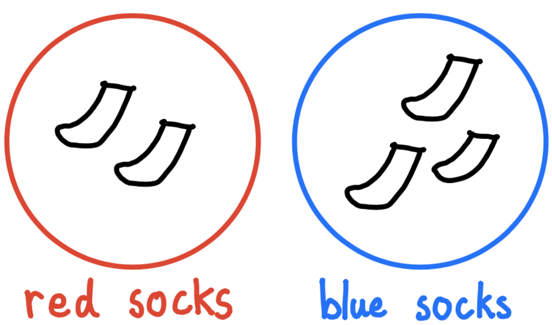
Example 13

Suppose that I have a bag containing 50 red socks and 50 blue socks. Imagine that I pull out socks from the bag without looking. How many socks do I have to pull out in order to guarantee that there are 3 socks that have the same colour?

Just like in Example 1, there are two categories (red socks and blue socks).

If I pull out 5 socks, then there are more socks than 2 times the number of categories.

Using the Pigeonhole Principle (for Multiple Repeated Items), I then know that one of the categories must contain at least $2+1=3$ socks.



Out of the 5 socks that I pull out, 3 of them must have the same colour.

Let's look at a few more examples.

Example 14

There are 60 toys within 14 boxes. How many toys must one of the boxes contain?

Solution: In this case, there are $k = 60$ toys that go into $c = 14$ boxes. Therefore, one of the boxes must contain $\lceil \frac{60}{14} \rceil = \lceil 4.\overline{285714} \rceil = 5$ toys.

Exercise 15: A bowl contains 58 Smarties. There are 8 different colours of Smarties. What is the largest number of Smarties of the same colour that the bowl must contain?

Exercise 16: If there are 10 students in 3 teams, then what is the minimum size of the largest team?

Exercise 15: The number of items is $k=58$ Smarties.
The number of categories is $c=8$ colours.

By the Pigeonhole Principle, there are at least $\lceil \frac{k}{c} \rceil$ smarties of the same colour.

So there are at least $\lceil \frac{58}{8} \rceil = \lceil 7.25 \rceil = 8$ smarties of the same colour.

Exercise 16: The number of items is $k=10$ students.
The number of categories is $c=3$ teams.

So there are are at least $\lceil \frac{10}{3} \rceil = \lceil 3\frac{1}{3} \rceil = 4$ on one of the teams.

Example 17

A Mersenne prime is a prime number that is one less than a power of two. For example, 31 is a Mersenne prime because it can be written as $2^5 - 1$. All Mersenne primes but the number 3 end in either the digit 1 or 7. Suppose that Emily has a list of 33 different Mersenne primes. She tells you that none of them are 3.

How many numbers on Emily's list can you guarantee end in the same digit?

Solution: On Emily's list, there are 33 Mersenne primes which must end in one of $k = 2$ digits. We can guarantee that at least $\lceil \frac{33}{2} \rceil = 17$ of the numbers on Emily's list end in the same 2 digit.

Exercise 18

Except for 2 and 5, all prime numbers end in one of the digits 1, 3, 7, or 9. In a list of 600 prime numbers, each greater than 5, at least how many have the same last digit?[2]

Here is an example to consider that requires applying the Pigeonhole Principle multiple times.

Exercise 19: At a private school, 70 students take between them 11 different classes. The maximum class size is 15. Show that there are 3 classes each attended by at least 5 students.

Solution: Since there are $k = 70$ students taking between them $c = 11$ classes, then immediately there is a class with $\lceil \frac{70}{11} \rceil = \lceil 6\frac{4}{11} \rceil = 7$ students.

There is a maximum of 15 students in the first class. So there are $70 - 15 = 55$ students between the 10 classes left. Among these 10 classes, there is one class with $\lceil \frac{55}{40} \rceil = \lceil 5.5 \rceil = 6$ students.

There is a maximum of 15 students in the second class. So there are $55 - 15 = 40$ students between the 9 classes left. Among these 9 classes, there is one class with $\lceil \frac{40}{9} \rceil = \lceil 4\frac{2}{3} \rceil = 5$ students.

Thus, we have shown that there are exists 3 classes with at least 5 students.

[2] *Solution:* We can guarantee that at least $\lceil \frac{600}{4} \rceil = 150$ numbers in the list have the same last digit.